Clifford Space as the Arena for Physics

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ABSTRACT

A new theory is considered according to which extended objects in n-dimensional space are described in terms of multivector coordinates which are interpreted as generalizing the concept of centre of mass coordinates. While the usual centre of mass is a point, by generalizing the latter concept, we associate with every extended object a set of r-loops, $r = 0, 1, ..., n-1$, enclosing oriented $(r+1)$-dimensional surfaces represented by Clifford numbers called $(r+1)$-vectors or multivectors. Superpositions of multivectors are called polyvectors or Clifford aggregates and they are elements of Clifford algebra. The set of all possible polyvectors forms a manifold, called C-space. We assume that the arena in which physics takes place is in fact not Minkowski space, but C-space. This has many far reaching physical implications, some of which are discussed in this paper. The most notable is the finding that although we start from the constrained relativity in C-space we arrive at the unconstrained Stueckelberg relativistic dynamics in Minkowski space which is a subspace of C-space.

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1 Introduction

A space (in particular, spacetime) consists of points (events). But besides points there are also lines, surfaces, volumes, etc.. Description of such geometric objects has turned out to be very elegant if one employs multivectors which are the outer products of vectors [1]. Multivectors and their linear combinations, polyvectors, are elements of Clifford algebras. Since in physics we do not consider point particles only, but also extended objects, it appears natural to consider Clifford algebra as an arena in which physics takes place [2, 3, 4]. Instead of spacetime we thus consider a more general space, the so called *Clifford manifold* or *C-space*. This is a space of the oriented $(r+1)$-dimensional areas enclosed by r-loops. In this paper I will show that extended objects can be described by r-loops and that the r-loop coordinates are natural generalizations of the concept of the center of mass coordinates. Besides the center of mass velocity an extended object has also the area velocity, the volume velocity, etc. (called multivector or holographic velocities). We generalize the theory of relativity from Minkowski space M_4 to C-space and thus bring into the game the holographic velocities. Besides the speed of light a fundamental length L has to be introduced. If we take L equal to the Planck length we find that the maximum holographic speeds are very slow and this explains why on the macroscopic scale we do not observe them. For instance, the volume (the 3-vector) speed is of the order of $10^{-62} m^3/sec$.

The action for a "point particle" in C-space is analogous to the action for a point particle in Minkowski space M_4. It is equal to the length of the world line in C-space. This action constrains the polymomentum to the mass shell in C-space. If we reduce the C-space action with respect to the 4-volume (4-vector or pseudoscalar) variable $s = X^{0123}$, then all other variables are independent and evolve with respect to s which assumes the role of evolution parameter (the true time). The action so reduced is equivalent to the well known Stueckelberg action of the relativistic dynamics [5].

In the unconstrained theory, minimal length, action, the variables $X^A = (\sigma, X^\mu, X^{\mu\nu}, X^{\mu\nu\alpha}, X^{\mu\nu\alpha\beta})$ are functions of an arbitrary parameter τ. The 4-volume also changes with τ. This explains why the world lines (actually the world tubes, if particles are extended) in M_4 are so long along time-like directions, and have so narrow space-like extension. This is just a very natural "final" state of objects evolving in M_4. Initially the objects may have arbitrary shapes, but if their 2-vector and 3-vector speeds are of the right sign (so it is on average in half of the cases), then their extensions along time-like directions will necessarily increase for positive 4-vector speeds; increasing 4-volume necessarily implies increasing length of a world tube (whose effective 2-area and 3-volume are constant or decreasing). Long world lines are necessary in order to provide the observed electromagnetic fields. Finite extensions of world lines provide a cutoff to the electromagnetic interaction which is predicted to change with time.

All the conservation laws are still valid, since the true physics is now in C-space. In C-space there is no "flow of time" at all: past, present and future coexist in the 16-dimensional "block" C-space, with objects corresponding to wordlines in C-space. But, if the 4-dimensional Minkowski space M_4 is considered as a slice which moves through C-space, objects on M_4 are evolving with respect to the Lorentz invariant evolution parameter s. There is then a genuine dynamics on M_4 which is induced by the postulated motion of M_4 through C-space. A reconsiliation between two seemingly antagonistic views is achieved, namely between the assertion that there is no time, no flow of time, etc., and the view that there is evolution, passage of time, relativistic dynamics. Both groups of thinkers are right, but each in its own space, and the link between the two views is in the postulated motion of M_4. So far an important argument against relativistic dynamics has been that it implies the entire spacetime moving in a 5-dimensional space and thus it introduces a "meta time". In this respect relativistic dynamics has been considered no better than the usual relativity together with the assumption that a "time slice" moves through M_4. Why to introduce a 5-dimensional space[1], and why should the game stop at five and not at six or more dimensions? Why not just stay with 4-dimensions of Minkowski space? Clifford space resolves those puzzles, since Clifford space is defined over 4-dimensional spacetime, it does not introduce extra dimensions of spacetime.

2 Geometry of spacetime and Clifford algebra

Usually it is assumed that physics takes place in the ordinary spacetime V_n, a manifold whose points are assigned the coordinates x^μ, $\mu = 0, 1, 2, ..., n-1$. Generally a spacetime or any manifold can be elegantly described by a set of basis vectors γ_μ satisfying the

[1]We are here not talking about the extra dimensions required by Kaluza-Klein theories or string theories, but about an extra dimensions introduced in order to describe evolution.

Clifford algebra relations

$$\gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu} \tag{1}$$

where $g_{\mu\nu}$ is the metric. In general γ_μ and thus $g_{\mu\nu}$ can depend on x^μ (see e.g., refs. [1, 4]). In terms of γ_μ a vector a in V_n can be expressed as

$$a = a^\mu \gamma_\mu \tag{2}$$

Its square is then

$$a^2 = a^\mu a^\nu \gamma_\mu \gamma_\nu = a^\mu a^\nu g_{\mu\nu} \equiv a^\mu a_\mu \tag{3}$$

A vector can in general be a field $a = a(x) = a^\mu(x)\gamma_\mu$ which depends on position. In particular, we may consider such a field $a(x)$ whose components are position coordinates:

$$a(x) = x = x^\mu \gamma_\mu \tag{4}$$

Acting on x by the differential operator d which acts on components, but leave the basis vectors unchanged (see [1, 4]) we obtain

$$\mathrm{d}x = \mathrm{d}x^\mu \gamma_\mu \tag{5}$$

This is an example of a *vector* in spacetime. Another example is the velocity $v = (\mathrm{d}x^\mu/\mathrm{d}\tau)\gamma_\mu$.

The well known important quantity is the square

$$\mathrm{d}x^2 = (\mathrm{d}x^\mu \gamma_\mu)^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{6}$$

which is the infinitesimal distance between two infinitesimally separated points ("events") in V_n.

Another quantity is the square of the coordinate vector field

$$a^2(x) = x^2 = (x^\mu \gamma_\mu)^2 = g_{\mu\nu} x^\mu x^\nu \tag{7}$$

which has a significant role in special relativity which acts in *flat* spacetime with the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$.

In a manifold we do not have points only. There are also lines, 2-dimensional surfaces, 3-dimensional surfaces, etc.. Description of all those objects requires use of higher degree vectors, i.e., *multivectors*, such as

$$\gamma_\mu \wedge \gamma_\nu \equiv \tfrac{1}{2!}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \equiv \tfrac{1}{2!}[\gamma_\mu, \gamma_\nu]$$

$$\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\alpha \equiv \tfrac{1}{3!}[\gamma_\mu, \gamma_\nu, \gamma_\alpha] \tag{8}$$

$$\text{etc.} \tag{9}$$

An *oriented line element* is given by a vector $\mathrm{d}x$ (eq.(5)).

An *oriented area element* is given by a bivector which is the wedge product of two different vectors $\mathrm{d}x_1$ and $\mathrm{d}x_2$:

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \mathrm{d}x_1^\mu \mathrm{d}x_2^\nu \, \gamma_\mu \wedge \gamma_\nu \tag{10}$$

Similarly for an arbitrary multivector

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge ... \wedge \mathrm{d}x_r = \mathrm{d}x_1^{\mu_1} ... \mathrm{d}x_r^{\mu_r} \, \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_r} \tag{11}$$

If r is less then the dimension of the space V_n, then the above multivector represents an r-dimensional *area element*. If $r = n$, then (11) becomes a *volume element* in the space V_n

$$\mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_n = \mathrm{d}x_1^{\mu_1} ... \mathrm{d}x_n^{\mu_n} \gamma \tag{12}$$

where the n-vector

$$\gamma \equiv \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_n} \tag{13}$$

is called *pseudoscalar*. It is the highest possible r-vector in a space of dimension n, since and $(n+1)$-vector is identically zero.

3 Multivectors associated with finite r-surfaces

Let us now consider $\mathrm{d}x$ of eq.(5) as the tangent vector to a curve $X^\mu(\tau)$. Integrating over the curve we have after a suitable choice of the integration constant that

$$\int \mathrm{d}x = \int \mathrm{d}\tau \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \gamma_\mu = x^\mu \gamma_\mu \tag{14}$$

where we have considered a *flat spacetime* and such a coordinate system in which γ_μ are constants. The result of integration in eq.(14) is independent of the chosen curve $X^\mu(\tau)$. It depends only on the initial and final point. In (14) we have taken the initial point at the coordinate origin and assigned to the final point coordinates x^μ.

Similarly we can consider $\mathrm{d}x_1 \wedge \mathrm{d}x_2$ as a tangent bivector to a surface $X^\mu(\sigma^1, \sigma^2)$. After the integration over a chosen range of the parameters $\sigma^1,\ \sigma^2$ we obtain

$$\begin{aligned}
\int \mathrm{d}x_1 \wedge \mathrm{d}x_2 &= \int \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \, \frac{\partial X^\mu}{\partial \sigma^1} \frac{\partial X^\nu}{\partial \sigma^2} \gamma_\mu \wedge \gamma_\nu \\
&= \frac{1}{2} \int \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \left(\frac{\partial X^\mu}{\partial \sigma^1} \frac{\partial X^\nu}{\partial \sigma^2} - \frac{\partial X^\nu}{\partial \sigma^1} \frac{\partial X^\mu}{\partial \sigma^2} \right) \gamma_\mu \wedge \gamma_\nu \\
&= \tfrac{1}{2} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu
\end{aligned} \tag{15}$$

where

$$X^{\mu\nu} = \int \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \left(\frac{\partial X^\mu}{\partial \sigma^1} \frac{\partial X^\nu}{\partial \sigma^2} - \frac{\partial X^\nu}{\partial \sigma^1} \frac{\partial X^\mu}{\partial \sigma^2} \right) \tag{16}$$

By the Stokes theorem this is equal to

$$X^{\mu\nu} = \frac{1}{2} \oint \mathrm{d}\zeta \left(X^\mu \frac{\partial X^\nu}{\partial \zeta} - X^\nu \frac{\partial X^\mu}{\partial \zeta} \right) \tag{17}$$

where the integral runs along a loop given by the embedding functions $X^\mu(\zeta)$ of a single parameter ζ.

The inegral (15) is a finite bivector and it does not depend on choice of the surface enclosed by the loop. There is a family of loops which all give the same result $X^{\mu\nu}$. Within the family there exists a subset of loops with all their points situated on a plane. Amongst them we may choose a representative loop. We then interprete $X^{\mu\nu}$ as bivector coordinates of a representative loop enclosing a flat surface element.

Choosing a coordinate system on the surface such that $\sigma^1 = X^1,\ \sigma^2 = X^2$ we find —when our surface is a flat rectangle— that

$$X^{12} = \int \mathrm{d}X^1 \, \mathrm{d}X^2 = X^1 \, X^2$$

$$X^{21} = - \int \mathrm{d}X^1 \, \mathrm{d}X^2 = -X^1 \, X^2 \tag{18}$$

Similarly, if we choose $\sigma^1,\ \sigma^2$ equal to some other pair of coordinates $X^\mu,\ X^\nu$, we obtain other components of $X^{\mu\nu}$ expressed as the products of X^μ and X^ν. In 4-dimensional spacetime we have

$$X^{\mu\nu} = \begin{pmatrix} 0 & X^0 X^1 & X^0 X^2 & X^0 X^3 \\ -X^1 X^0 & 0 & X^1 X^2 & X^1 X^3 \\ -X^2 X^0 & -X^2 X^1 & 0 & X^2 X^3 \\ -X^3 X^0 & -X^3 X^1 & -X^3 X^2 & 0 \end{pmatrix} \tag{19}$$

These are just the holographic projections of our oriented surfaces onto the coordinate planes.

In analogous way we can calculate higher multivector components $X^{\mu_1\mu_2...\mu_r}$ for arbitrary $r \leq n$.

A *vector* $x \equiv X_1 = X^\mu \gamma_\mu$ can be used to denote a *point*, and X^μ are *coordinates* of the point. The *distance* of the point from the coordinate origin is given by the square root of the expression (7). In other words, the *length* of line from the coordinate origin to the point is $(g_{\mu\nu} X^\mu X^\nu)^{1/2}$, and the length square in flat spacetime is

$$|X_1|^2 \equiv X_1 * X_1 = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 \tag{20}$$

Analogously, a bivector $X_2 = \frac{1}{2} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$ denotes a 1-*loop* enclosing a flat surface element, and $X^{\mu\nu}$ are (bivector type) coordinates of the loop. The area that the loop encloses is given by the square root of the scalar product between $X_2 = \frac{1}{2} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$ and its Hermitian conjugate or reverse $X_2^\dagger = \frac{1}{2} X^{\mu\nu} \gamma_\nu \wedge \gamma_\mu$:

$$\begin{aligned} X_2^\dagger * X_2 &= \tfrac{1}{4} [X^{\mu\nu} \gamma_\nu \wedge \gamma_\mu] * [X^{\alpha\beta} \gamma_\alpha \wedge \gamma_\beta] \\ &= \tfrac{1}{4} X^{\mu\nu} X^{\alpha\beta} (\gamma_\nu \wedge \gamma_\mu) \cdot (\gamma_\alpha \wedge \gamma_\beta) \\ &= \tfrac{1}{4} X^{\mu\nu} X^{\alpha\beta} \left[(\gamma_\mu \cdot \gamma_\alpha)(\gamma_\nu \cdot \gamma_\beta) - (\gamma_\mu \cdot \gamma_\beta)(\gamma_\nu \cdot \gamma_\alpha) \right] \\ &= \tfrac{1}{4} X^{\mu\nu} X^{\alpha\beta} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) = \tfrac{1}{2} X^{\mu\nu} X_{\mu\nu} \end{aligned} \tag{21}$$

In a four-dimensional Minkowski space with signature $(+ - --)$ we have

$$|X_2|^2 = -(X^{01})^2 - (X^{02})^2 - (X^{03})^2 + (X^{12})^2 + (X^{13})^2 + (X^{23})^2 \tag{22}$$

which is the Pythagora rule for surfaces, analogous to the one for lines.

An r-*vector* $X_r = \frac{1}{r!} X^{\mu_1...\mu_r} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \gamma_{\mu_r}$ denotes an oriented r-*surface* enclosed by an $(r-1)$-*loop*, where $X^{\mu_1...\mu_r}$ are (r-vector type) *coordinates* of the r-surface.

The precise shape of the $(r-1)$-loop is not determined by the r-vector coordinates $X^{\mu_1...\mu_r}$. Only the orientation and the *area* enclosed by the loop are determined. These are thus *collective coordinates* of a loop. They do not describe all the degrees of freedom of a loop, but only its collective degree of freedom—area and orientation— common to a family of loops.

Multivectors are elements of the *Clifford algebra* generated by the set of basis vectors $\gamma_{\mu_1}, ..., \gamma_{\mu_r}$. A generic *Clifford number* or *polyvector* (also called Clifford aggregate) is a sum of multivectors:

$$X = \sigma + X^\mu \gamma_\mu + \frac{1}{2!} X^{\mu_1 \mu_2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} + ... + \frac{1}{n!} X^{\mu_1 ... \mu_n} \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_n} \tag{23}$$

where $X^A = (\sigma, X^\mu, X^\mu, ...)$ and $E_A = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu}, ...)$.

Taking $n = 4$ and using the relations

$$\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma = I \epsilon_{\mu\nu\rho\sigma} \tag{24}$$

$$\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho = I \epsilon_{\mu\nu\rho\sigma} \gamma^\sigma \tag{25}$$

$$I \equiv \gamma = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \tag{26}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor, and introducing the new coefficients

$$\xi_\sigma \equiv \frac{1}{3!} X^{\mu\nu\rho} \epsilon_{\mu\nu\rho\sigma} \; , \qquad s \equiv \frac{1}{4!} X^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \tag{27}$$

we can rewrite (23) in terms of pseudovector and pseudoscalar variables

$$X = \sigma + X^\mu \gamma_\mu + \frac{1}{2} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \xi^\mu I \gamma_\mu + sI \tag{28}$$

Summation of multivectors of different degrees r is no more curious than the summation of real and imaginary numbers. The set of all possible polyvectors X forms a manifold, called *Clifford space*, or briefly, *C-space*.

4 Multivectors associated with physical objects

We will now show that the r-vector coordinates $X^{\mu_1 ... \mu_r}$ when considered as describing a physical object—not necessarily a loop—are a generalization of the *centre of mass coordinates* defined by the weighted sum (or integral) over the position vectors of matter distribution within the object. Assuming that the object—which exists in V_n—consists of discrete events with coordinates X_i^μ, all having the same weight, we have

$$X^\mu = \frac{1}{N} \sum_i X_i^\mu \tag{29}$$

where N is the number of events. The events are assigned the same weight $1/N$ if they have the same masses. In general masses m_i are different and we have

$$X^\mu = \sum_i X_i^\mu \rho_i \tag{30}$$

where $\rho_i \equiv m_i / \sum_j m_j$ is the weight associated to the i-th event. The centre of mass coordinates X^μ are collective *vector coordinates* associated with the object: they determine the "effective" position of the object.

Let us now generalize the concept of weight by introducing the *two point weights* ρ_{ij} which are functions not of a single point, but of two points within an object. Then we can define the collective *bivector coordinates* $X^{\mu\nu}$ associated with an object:

$$X^{\mu\nu} = \sum_{ij} X_i^\mu X_j^\nu \rho_{ij} \tag{31}$$

They determine the effective 2-vector orientation and area of the object. In other words, $X^{\mu\nu}$ determine how much the object—with a given ρ_{ij}—deviates from a point like object: they determine a fictitious 1-loop enclosing a 2-area defined by the bivector $X^{\mu\nu}$.

In general, the collective r-vector coordinates $X^{\mu_1 ... \mu_r}$, $r \leq n$, associated with an object in an n-dimensional space we define as

$$X^{\mu_1 ... \mu_r} = \sum_{i_1, i_2, ..., i_r} X_{i_1}^{\mu_1} X_{i_2}^{\mu_2} ... X_{i_r}^{\mu_r} \rho_{i_1 i_2 ... i_r} \tag{32}$$

They determine a fictitious $(r-1)$-loop which is a measure of the object's effective r-volume. In particular, the considered objects may actually be an $(r-1)$-loop, but in general it need not be: and yet, using the definition (32) we still associate with the object an $(r-1)$-loop, which can be called a *centre of mass* $(r-1)$*-loop*. The corresponding 0-loop is the ordinary *centre of mass*.

In the usual dynamics (relativistic or non relativistic) we are used to describing the extended objects—which are not strings or the Dirac-Nambu-Goto branes— as being point particles sitting in the object centre of mass. Thus we take into account the information about the motion of the object's centre of mass, but neglect the extension of the object.

In general, of course, the object's *extension* also should be included into a dynamical description. The object's r-vector coordinates (32) provide a natural means of how to take into account the fact that the object is not a point particle but has a finite extension.

4.1 Centre of mass polyvector

The above description was just to introduce the concept of r-vector coordinates of a physical object. We will now further formalize this by introducing the *centre of mass* *polyvector* X according to

$$X = \rho \underline{1} + \int \mathrm{d}^n x \, \rho(x) x^\mu \gamma_\mu + \frac{1}{2!} \int \mathrm{d}^n x \, \mathrm{d}^N x' \, \rho(x, x') x^\mu x'^\mu \, \gamma_\mu \wedge \gamma_\nu$$
$$+ \frac{1}{3!} \int \mathrm{d}^n x \, \mathrm{d}^N x' \, \mathrm{d}^N x'' \, \rho(x, x', x'') x^\mu x^\nu x^\alpha \, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\alpha + ... \qquad (33)$$

Here $\rho(x)$ is the ordinary mass density, normalized according to $\int \rho(x) \mathrm{d}^n x = 1$, whilst ρ, $\rho(x, x')$, $\rho(x, x', x'')$,..., are the corresponding 0-point, 2-point, 3-point, ..., generalizations. Namely, we generalize the concept of mass density so that besides the usual mass density $\rho(x)$ which is a 1-point function, we introduce also "mass" densities which are functions of any number of points.

We can distinguish two particular cases:

(i) Mass densities have support on a discrete set of spacetime points:

$$\begin{aligned} \rho(x) &= \sum_i \rho_i \, \delta(x - X_i) \\ \rho(x, x') &= \sum_{ij} \rho_{ij} \, \delta(x - X_i) \delta(x - X_j) \\ &\vdots \end{aligned} \qquad (34)$$

Then the centre of mass polyvector (33) becomes

$$X = \rho \underline{1} + \sum_i \rho_i \, X_i^\mu \gamma_\mu + \frac{1}{2!} \sum_{ij} \rho_{ij} \, X_i^\mu X_j^\nu \gamma_\mu \wedge \gamma_\nu + ... \qquad (35)$$

Since, on the other hand X can be expanded according to eq.(23) its multivector components are given by (29)–(32). The scalar component is ρ.

(ii) Mass densities have support on a brane described by embedding functions $X^\mu(\xi) \equiv X^\mu(\xi^a)$, $\mu = 0, 1, ..., n-1; a = 0, 1, ..., p:$

$$\begin{aligned} \rho(x) &= \int \mathrm{d}\xi \, \rho(\xi) \, \delta(x - X(\xi)) \\ \rho(x, x') &= \int \mathrm{d}\xi \, \mathrm{d}\xi' \, \rho(\xi, \xi') \delta(x - X(\xi)) \delta(x' - X(\xi')) \\ &\vdots \end{aligned} \qquad (36)$$

Then the centre of mass polyvector (33) is

$$X = \rho \, \underline{1} + \int \mathrm{d}\xi \, \rho(\xi) \, X^\mu(\xi) \gamma_\mu + \frac{1}{2!} \int \mathrm{d}\xi \, \mathrm{d}\xi' \, \rho(\xi, \xi') \, X^\mu(\xi) X^\nu(\xi') \gamma_\mu \wedge \gamma_\nu + ... \qquad (37)$$

(iii) Mass densities have support on a set of branes of various dimensionalities, each brane being described by the embedding functions $X_i^\mu(\xi) \equiv X_i^\mu(\xi^{a_i}),\ \mu = 0,1,2,...,n-1;\ a_i = 0,1,2,...,p_i$:

$$
\begin{aligned}
\rho(x) &= \sum_i \int \mathrm{d}\xi \, \rho_i(\xi) \, \delta(x - X_i(\xi)) \\
\rho(x, x') &= \sum_{ij} \int \mathrm{d}\xi \, \mathrm{d}\xi' \, \rho_{ij}(\xi, \xi') \, \delta(x - X_i(\xi)) \delta(x' - X_j(\xi')) \\
&\vdots
\end{aligned}
\tag{38}
$$

Then the centre of mass polyvector (33) assumes the form

$$
X = \rho \underline{1} + \sum_i \int \mathrm{d}\xi \, \rho_i(\xi) X_i^\mu(\xi) \gamma_\mu + \frac{1}{2!} \sum_{ij} \int \mathrm{d}\xi \, \mathrm{d}\xi' \, \rho_{ij}(\xi, \xi') X_i^\mu(\xi) X_j^\nu(\xi') \gamma_\mu \wedge \gamma_\nu + ... \tag{39}
$$

For simplicity we have omitted the subscript i on the i-th brane parameters $\xi \equiv \xi^{a_i}$.

Suppose now that our system consists of branes whose sizes can be neglected at large distances so that approximately we have

$$
\rho_i(\xi) = \rho_i \delta(\xi - \bar{\xi}) \, , \qquad \rho_{ij}(\xi, \xi') = \rho_{ij} \delta(\xi - \bar{\xi}) \delta(\xi' - \bar{\xi}) \tag{40}
$$

Then the expression (39) can be approximated by the expression (35). The latter expression (35) has non vanishing r-vector part, $r = 2, 3, ...$, when $\rho_{ij},\ \rho_{ijk}$,..., are different from zero and are not symmetric in i, j..., i.e., they contain the antisymmetric parts. Then our system is anisostropic, owing to the asymmetric 2-point, 3-point,..., mass densities $\rho_{ij},\ \rho_{ijk}, ...$, and this manifests itself in non vanishing multivector centre of mass coordinates $X^{\mu\nu},\ X^{\mu\nu\alpha}, ...$ However, we may envisage the situation in which $\rho_{ij},\ \rho_{ijk}, ...$ are symmetric in their indices. Then our system—neglecting the fact that its constituents are not point particles, but branes—is *isotropic*, and consequently $X^{\mu\nu},\ X^{\mu\nu\alpha}, ...$ are zero.

Now suppose that we look at our system more closely and take into account that it consists of branes. Even if $\rho_{ij}(\xi, \xi')$ in eq. (39) is symmetric in the indices i, j, the bivector term in (39) can be different from zero if there is antisymmetry in the parameters ξ, ξ'. For instance, if we assume that

$$
\rho_{ij}(\xi, \xi') = \delta_{ij} \rho_j(\xi, \xi') \tag{41}
$$

where $\rho_i(\xi, \xi') = -\rho_i(\xi', \xi)$, then the bivector part of the centre of mass polyvector (39) then becomes

$$
\langle X \rangle_2 = \frac{1}{2} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu = \frac{1}{2} \sum_i \int \mathrm{d}\xi \, \mathrm{d}\xi' \, \rho_i(\xi, \xi') X_i^\mu(\xi) X_i^\nu(\xi') \gamma_\mu \wedge \gamma_\nu \tag{42}
$$

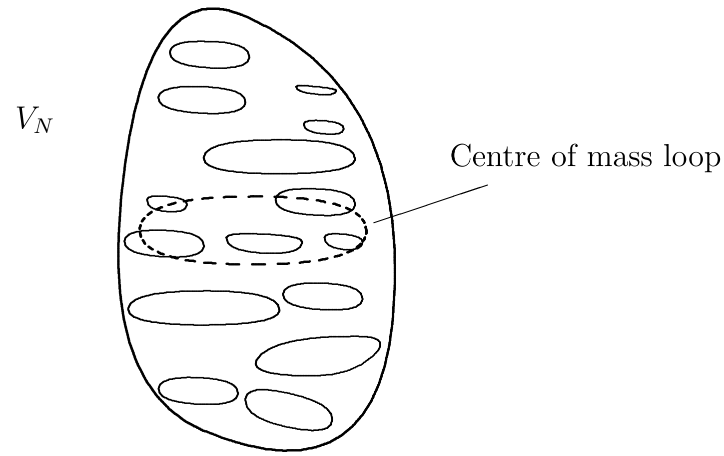

Figure 1: If a physical object is a system of 1-loops with a prefered orientation, then we can associate with the system an "effective" or *centre of mass 1-loop*.

from which we have

$$X^{\mu\nu} = \frac{1}{2} \sum_i \int \mathrm{d}\xi \, \mathrm{d}\xi' \, \rho_i(\xi, \xi')(X_i^\mu(\xi) X_i^\nu(\xi') - X_i^\mu(\xi') X_i^\nu(\xi)) \tag{43}$$

Suppose that all the branes within the system are closed 1-branes, i.e., 1-loops. If there are many branes and if they are randomly oriented, then the sum in eq.(43) results in $X^{\mu\nu} \approx 0$. But, if the branes have a prefered orientation, then the centre of mass bivector coordinates $X^{\mu\nu}$ differ from zero and they determine an "effective" or *centre of mass 1-loop* associated with our system of loops (branes) (see Fig. 1).

4.2 A special case: the extended object is a single loop

Let us now consider a special case in which our extended object is a single 1-loop parametrized by ξ. Eq. (43) then reads

$$X^{\mu\nu} = \frac{1}{2} \int \mathrm{d}\xi \, \mathrm{d}\xi' \, \rho(\xi, \xi')(X^\mu(\xi) X^\nu(\xi') - X^\mu(\xi') X^\nu(\xi)) \tag{44}$$

where the integral runs over the (closed) loop so that the symbol $\int$ can be replaced by $\oint$. We shall now demonstrate that for a suitably chosen $\rho(\xi, \xi')$ the expression (44) for $X^{\mu\nu}$ is identical to the expression (16) which determines the area enclosed by the loop. We shall consider two possible choices of $\rho(\xi, \xi')$.

According to the *first choice* we set

$$\rho(\xi, \xi') = \frac{\partial}{\partial \xi} \delta(\xi - \xi') \tag{45}$$

Inserting this into eq.(44) we obtain after integrating out ξ' the expression

$$X^{\mu\nu} = \frac{1}{2} \oint \mathrm{d}\xi \left(X^\mu(\xi) \frac{\partial X^\nu(\xi)}{\partial \xi} - X^\nu(\xi) \frac{\partial X^\mu(\xi)}{\partial \xi} \right) \tag{46}$$

which—according to the Stokes theorem—is equivalent to the expression (16) for the oriented area of enclosed by the loop.

So we have shown that a loop can be considered as an extended object, with the two point density (45), whose *centre of mass bivector coordinates* $X^{\mu\nu}$ —calculated according to eq.(44) (which in turn is a part of the general formula (33))— are just the projections of the oriented area enclosed by the loop. By this we have checked the consistency of the definition (33) for the centre of mass bivector coordinates $X^{\mu\nu}$. The analog is expected to be true for higher multivector coordinates $X^{\mu\nu\alpha\dots}$.

According to *the second choice* we can set

$$\rho(\xi, \xi') = \frac{1}{2}\, \epsilon(\xi, \xi') \tag{47}$$

where $\epsilon(\xi, \xi') = -\epsilon(\xi', \xi) = 1$. Then in the case in which the loop is a circle we have

$$\begin{aligned} X^{\mu\nu} &= \frac{1}{4} \int \mathrm{d}\varphi \mathrm{d}\varphi'\, \epsilon(\varphi, \varphi')(X^\mu(\varphi) X^\nu(\varphi') - X^\nu(\varphi) X^\mu(\varphi')) \\ &= \frac{1}{2} \int_0^{2\pi} \mathrm{d}\varphi \int_\varphi^{2\pi} \mathrm{d}\varphi' (X^\mu(\varphi) X^\nu(\varphi') - X^\nu(\varphi) X^\mu(\varphi')) \end{aligned} \tag{48}$$

Taking the parametric equation of the circle, namely $X^1 = r\cos\varphi$, $X^2 = r\sin\varphi$, we find that

$$X^{12} = \frac{r^2}{2} \int_0^{2\pi} \mathrm{d}\varphi \int_\varphi^{2\pi} \mathrm{d}\varphi' \,(\cos\varphi \sin\varphi' - \sin\varphi\cos\varphi') = \frac{r^2}{2} \int_0^{2\pi} \mathrm{d}\varphi (-\cos\varphi + 1) = \pi r^2 \tag{49}$$

which is just the area of the circle. It would be interesting to investigate what eq.(48) would give for an arbitrary closed curve $X^\mu(\varphi)$.

5 Relativity in C-space

5.1 Kinematics

All r-vector coordinates of a physical object are in principle different from zero, and they all have to be taken into account. Vectors $X^\mu \gamma_\mu$, bivectors $\frac{1}{2!} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$, and in general r-vectors (or multivectors) $\frac{1}{r!} X^{\mu_1 \dots \mu_r} \gamma_{\mu_1} \wedge \dots \wedge \gamma_{\mu_r}$ they all enter the description of an object.

Compact notation A coordinate polyvector can be written as

$$X = \frac{1}{r!} \sum_{r=0}^{N} X^{\mu_1 ... \mu_r} \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_r} \equiv X^A E_A \tag{50}$$

Here we use a compact notation in which $X^A \equiv X^{\mu_1 ... \mu_r}$ are coordinates, and $E_A \equiv \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_r}$, $\mu_1 < \mu_2 < ... < \mu_r$, basis vectors of the 2^n-dimensional *Clifford algebra* of spacetime. The latter algebra of spacetime positions and corresponding higher grade objects, namely oriented r-areas, is a manifold which is more general than spacetime. In the literature such a manifold has been named *pandimensional continuum* [2] or *Clifford space* or *C-space* [3].

The infinitesimal element of position polyvector (12) is

$$\mathrm{d}X = \frac{1}{r!} \sum_{r=0}^{N} \mathrm{d}X^{\mu_1 ... \mu_r} \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_r} \equiv \mathrm{d}X^A E_A \tag{51}$$

We will now calculate its square. Using the definition for the *scalar product* of two polyvectors A and B

$$A * B = \langle AB \rangle_0 \tag{52}$$

where $\langle \ \rangle_0$ means the scalar part of the geometric product AB, we obtain

$$|\mathrm{d}X|^2 \equiv \mathrm{d}X^\dagger * \mathrm{d}X = \mathrm{d}X^A \, \mathrm{d}X^B G_{AB} = \mathrm{d}X^A \, \mathrm{d}X_A \tag{53}$$

Here

$$G_{AB} = E_A^\dagger * E_B \tag{54}$$

is the C-space metric and $A^\dagger$ the reverse[2] of a polyvector A.

For example, if the indices assume the values $A = \mu$, $B = \nu$, we have

$$G_{\mu\nu} = \langle e_\mu e_\mu \rangle_0 = e_\mu \cdot e_\nu = g_{\mu\nu} \tag{55}$$

If $A = [\mu\nu]$, $B = [\alpha\beta]$

$$\begin{aligned} G_{[\mu\nu][\alpha\beta]} &= \langle (e_\mu \wedge e_\nu)^\dagger (e_\alpha \wedge e_\beta) \rangle_0 = \langle (e_\mu \wedge e_\nu)^\dagger \cdot (e_\alpha \wedge e_\beta) \rangle_0 \\ &= (e_\mu \cdot e_\alpha)(e_\nu \cdot e_\beta) - (e_\nu \cdot e_\alpha)(e_\mu \cdot e_\beta) = g_{\mu\alpha} g_{\nu\beta} - g_{\nu\alpha} g_{\mu\beta} \end{aligned} \tag{56}$$

If $A = \mu$, $B = [\alpha\beta]$

$$G_{\mu[\alpha\beta]} = \langle e_\mu (e_\alpha \wedge e_\beta) \rangle_0 = 0 \tag{57}$$

[2]Reversion or *hermitian conjugation* is the operation [1] which reverses the order of all products of vectors in a decomposition of a polyvector A. For instance, $(\gamma_1 \gamma_2 \gamma_3)^\dagger = \gamma_3 \gamma_2 \gamma_1$.

Explicitly we have

$$
\begin{aligned}
|\mathrm{d}X|^2 &= \frac{1}{r!}\sum_{r=0}^{N} \mathrm{d}X^{\mu_1...\mu_r}\mathrm{d}X_{\mu_1...\mu_r} \\
&= \mathrm{d}\sigma^2 + \mathrm{d}X^\mu \mathrm{d}X_\mu + \frac{1}{2!}\mathrm{d}X^{\mu_1\mu_2}\mathrm{d}X_{\mu_1\mu_2} + ... + \frac{1}{n!}\mathrm{d}X^{\mu_1...\mu_n}\mathrm{d}X_{\mu_1...\mu_n} \qquad (58)
\end{aligned}
$$

In C-space the usual points, lines, surfaces, etc., are all described on the same footing and can be transformed into each other by rotations in C-space (polydimensional rotations):

$$X'^A = L^A{}_B X^B \qquad (59)$$

These are generalizations of the usual Lorentz transformations. They mix multivectors of different grades. For instance, a pure vector can acquire a bivector component, so that after the transformation it is a superposition of a vector and a bivector. This means that a point with coordinates $X^A = (0, x^\mu, 0, 0, 0)$ can become a loop with coordinates $X'^A = (0, x'^\mu, x'^{\mu\nu}, 0, 0)$.

The generalized Lorentz transformations, i.e., the Lorentz transformations in C-space, include the ordinary Lorentz boosts which connect the reference systems in relative translational motion. In addition they also include the boost which connect the reference frames in relative dilatational motion. All experiences that we have from the relativity in Minkowski space are directly applicable to C-space. The line element, the action, the transformations between reference frames have the same form, only the set of variables (and its interpretation) is different. Instead of x^μ we have X^A, which are the holographic projections of the centre of mass $(r-1)$-loops.

5.2 Dynamics

The dynamical variables of our physical object are given by a polyvector X. The *action* is a generalization of the point particle action:

$$I = \kappa \int \mathrm{d}\tau\, (\dot{X}^\dagger * \dot{X})^{1/2} = \kappa \int \mathrm{d}\tau\, (\dot{X}^A \dot{X}_A)^{1/2} \qquad (60)$$

where κ is a constant, playing the role of "mass" in C-space, and τ is an arbitrary parameter.

The theory so obtained is an extension of the ordinary special relativity. We shall call it *extended relativity* or *relativity in C-space*.

The equation of motion resulting from (60) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{X}^A}{\sqrt{\dot{X}^B\dot{X}_B}}\right)=0 \tag{61}$$

Taking $\dot{X}^B\dot{X}_B = \text{constant} \neq 0$ we have that $\ddot{X}^A = 0$, so that $X^A(\tau)$ is a straight worldline in C-space. The components X^A then change linearly with the parameter τ.

Let us consider more closely what does it mean physically that the multivector components are linear functions of τ. This implies that the holographic projections $X^A \equiv X^{\mu_1\ldots\mu_r}$ of the centre of mass $(r-1)$-loops are linear functions of τ. This means that the orientation and the area of the r-surface enclosed by the $(r-1)$-loop can increase or decrease. In particular, the initial conditions can be such that the orientation remains the same and only the area changes. In such a case we have a pure *dilatational motion* (without rotational motion) of the $(r-1)$-loop. The loop changes its *scale* (i.e., its size, extension) during its motion (Fig. 2). The dilatational degree of freedom (briefly, *scale*) is encoded in the holographic coordinates $X^{\mu_1\ldots\mu_r}$ for $r \geq 2$. The theory here suggests that *scale* is a degree of freedom in the analogous way as the centre of mass position is a degree of freedom. Namely, by definition $X^{\mu\nu\ldots}$ are degrees of freedom, and we have seen that they determine the effective extension (size or scale) of the object.

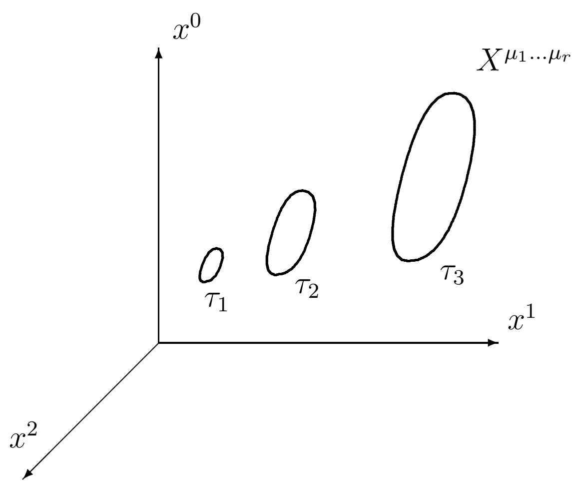

Figure 2: In general an $(r-1)$-loop changes with τ. Its *orientation* and *size* increases or decreases. The picture illustrates the *dilatational motion* of a loop.

So we have discovered that *scale* as a degree of freedom of an object is inherent in the Clifford algebra extension of the usual point particle kinematics and dynamics. The dynamics in C-space is more general than it is usually assumed. In the usual dynamics the scale of a free rigid object is fixed; it cannot change. Only the object's centre of mass position can change. In the C-space dynamics the scale of a free object can change. The scaled object remains similar to the original object, only all the distances and other scale dependent quantities within or related to the object are changed by a scale factor. Such a scale change should be understood as to be applicable to all sorts of objects and its constituents, the atoms, nuclei, etc. The theory predicts *scaled atoms* with scaled spectral lines. A possible objection, namely that quantum mechanics does not allow this, is not valid here. The ordinary quantum mechanics takes into account the position X^μ and position dependent forces between the objects (i.e., nuclei and electrons) constituting the atoms and so it comes out that atoms have fixed sizes. If we generalize quantum mechanics so to include the holographic coordinates $X^{\mu\nu\ldots}$ into the description, the situation is different. Scale then becomes a degree of freedom and has to be quantized as well. Like position, the scale of a *free* object can be arbitrary, whilst the scale of a bound object has discrete quantum values. If the atom as a whole is a free object, its centre of mass position can be arbitrary. Similarly, its *scale* can be arbitrary. What is not arbitrary is relative positions of electrons within the atom: they are determined by the solutions of the Schrödinger equation (generalized to C-space). The translational and dilatational motion of the atom as a whole, i.e., how the centre of mass coordinate X^μ and the generalized centre of mass coordinates $X^{\mu\nu\ldots}$ move, also is determined by the Schrödinger equation in C-space. We shall not go into a more detailed description here (see refs. [4, 7]).

The objects's 4-volume X^{0123} also changes with τ. With increasing τ an arbitrary object in V_4 can become more and more similar to *world line* (Fig. 3).

So the physics in C-space predicts long, but not infinite, worldlines. This has consequences for the electromagnetic interaction.

5.3 The electromagnetic field

Although a complete treatment of the minimal coupling and gauge fields in C-space remains to be elaborated[3] we anticipate here that the electromagnetic potential A^μ around a finite worldline is

$$A^\mu(x) = \int_{-\ell/2}^{\ell/2} e\,\delta[(x - X(\lambda))^2]\dot{X}^\mu(\lambda)\,\mathrm{d}\lambda \tag{62}$$

[3]This is one of our next projects [8].

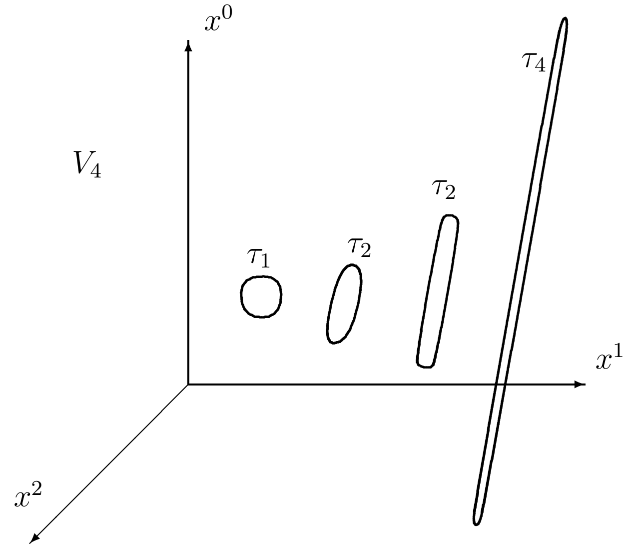

Figure 3: Illustration of 4-volume changes as a result of the dynamics in C-space. An object's 4-volume increases when its 4-vector speed is different from zero and positive. If the space-like components of its 3-vector and 2-vector velocity are zero or negative, the 4-volume can increase only if the object become more and more elongated in the time-like direction

Here $X^\mu(\lambda)$ are embedding functions of the worldline, paramterized by λ, e the electric charge and x^μ the coordinates of any point in the embedding spacetime (Fig. 4).

The ordinary electromagnetic interaction is obtained in the limit of infinitly long worldlines ($\ell \to \infty$). In this theory electromagnetic interaction depends on ℓ. Since ℓ changes with τ, electromagnetic interaction also changes. It remains as a future project to elaborate this and calculate numerically how electromagnetic interaction changes, and compare the result with the recent claims based, on astrophysical data, that the fine structure constant α has changed during the evolution of the Universe [9].

5.4 Estimation of holographic speeds

From the action (60) we find the following expression for the momentum in C-space:

$$P_A = \frac{\kappa \dot{X}_A}{(\dot{X}^B \dot{X}_B)^{1/2}} \tag{63}$$

When the denominator in eq.(3.2) is zero the momentum becomes infinite. We shall now calculate the speed at which this happens. This will be the *maximum speed* that an object

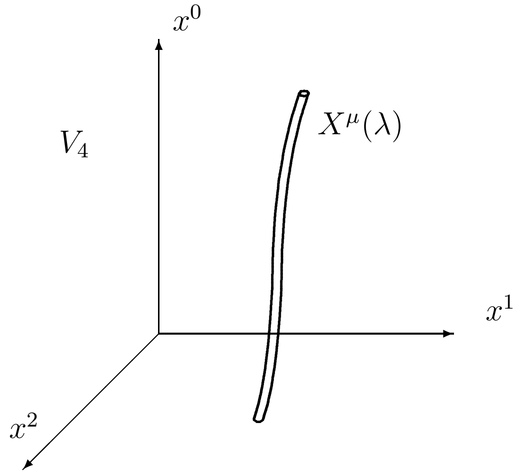

Figure 4: A finite, electrically charged, worldline-like object $\mathcal{O}$ of a finite length ℓ, produces at an arbitrary point x^μ the electromagnetic potential $A^\mu(x)$ which deviates from the ordinary electromagnetic potential. The deviation becomes smaller and smaller with increasing ℓ.

accelerating in C-space can reach[4].

Vanishing of $\dot{X}^B \dot{X}_B$ is equivalent to vanishing of the C-space line element

$$\mathrm{d}X^A \mathrm{d}X_A = \mathrm{d}\sigma^2 + \left(\frac{\mathrm{d}X^0}{L}\right)^2 - \left(\frac{\mathrm{d}X^1}{L}\right)^2 + \left(\frac{\mathrm{d}X^{12}}{L^2}\right)^2 - \left(\frac{\mathrm{d}X^{123}}{L^3}\right)^2 - \left(\frac{\mathrm{d}X^{0123}}{L^4}\right)^2 + ... = 0 \quad (64)$$

where by "..." we mean the terms with the remaining components such as X^2, X^{01}, X^{23},..., X^{012}, etc.. In eq. (64) we introduce a length parameter L. This is necessary, since $X^0 = ct$ has dimension of length, X^{12} of length square, X^{123} of length to the third power, and X^{0123} of length to the forth power. It is natural to assume that L is the *Planck length*, that is $L = 1.6 \times 10^{-35} m$.

Let us assume that the coordinate time $t = X^0/c$ is the parameter with respect to which we define the speed V in C-space. So we have

$$V^2 = -\left(L\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}X^1}{\mathrm{d}t}\right)^2 - \left(\frac{1}{L}\frac{\mathrm{d}X^{12}}{\mathrm{d}t}\right)^2 + \left(\frac{1}{L^2}\frac{\mathrm{d}X^{123}}{\mathrm{d}t}\right)^2 + \left(\frac{1}{L^3}\frac{\mathrm{d}X^{0123}}{\mathrm{d}t}\right)^2 - ... \quad (65)$$

The maximum speed is given by

$$V^2 = c^2 \quad (66)$$

[4]Although an initially slow object cannot accelerate beyond that speed limit, this does not automatically exclude the possibility that fast objects traveling at a speed above that limit may exist. Such objects are C-space analog of tachyons. All the well known objections against tachyons should be reconsidered for the case of C-space before we could say for sure that C-space tachyons do not exist as freely propagating objects.

First, we see that the maximum speed squared V^2 contains not only the components of the 1-vector velocity $\mathrm{d}X^1/\mathrm{d}t$, as it is the case in the ordinary relativity, but also the multivector components such as $\mathrm{d}X^{12}/\mathrm{d}t$, $\mathrm{d}X^{123}/\mathrm{d}t$, etc..

The following special cases when only certain components of the velocity in C-space are different from zero, are of particular interest:

(i) Maximum 1-vector speed

$$\frac{\mathrm{d}X^1}{\mathrm{d}t} = c = 3.0 \times 10^8 m/s$$

(ii) Maximum 3-vector speed

$$\frac{\mathrm{d}X^{123}}{\mathrm{d}t} = L^2 c = 7.7 \times 10^{-62} m^3/s$$
$$\frac{\mathrm{d}\sqrt[3]{X^{123}}}{\mathrm{d}t} = 4.3 \times 10^{-21} m/s \quad \text{(diameter speed)}$$

(iii) Maximum 4-vector speed

$$\frac{\mathrm{d}X^{0123}}{\mathrm{d}t} = L^3 c = 1.2 \times 10^{-96} \mathrm{m}^4/\mathrm{s}$$
$$\frac{\mathrm{d}\sqrt[4]{X^{0123}}}{\mathrm{d}t} = 1.05 \times 10^{-24} m/s \quad \text{(diameter speed)}$$

Above we have also calculated the corresponding diameter speeds for the illustration of how fast the obejct expands or contracts.

We see that the maximum multivector speeds are very small. The diameters of objects change very slowly. Therefore we normally do not observe the dilatational motion.

Because of the positive sign in front of the σ and X^{12}, X^{012}, etc., terms in the quadratic form (64) there are no limits to correspondintg 0-vector, 2-vector and 3-vector speeds. But if we calculate, for instance, the energy necessary to excite 2-vector motion we find that it is very high. Or equivalently, to the relatively modest energies (available at the surface of the Earth), the corresponding 2-vector speed is very small. This can be seen by calculating the energy

$$p^0 = \frac{\kappa c^2}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{67}$$

(a) for the case of pure 1-vector motion by taking $V = \mathrm{d}X^1/\mathrm{d}t$, and

(b) for the case of pure 2-vector motion by taking $V = \mathrm{d}X^{12}/(L\mathrm{d}t)$.

By equating the energies belonging to the cases (a) and (b we have

$$p^0 = \frac{\kappa c^2}{\sqrt{1 - \left(\frac{1}{c}\frac{\mathrm{d}X^1}{\mathrm{d}t}\right)^2}} = \frac{\kappa c^2}{\sqrt{1 - \left(\frac{1}{Lc}\frac{\mathrm{d}X^{12}}{\mathrm{d}t}\right)^2}} \tag{68}$$

which gives

$$\frac{1}{c}\frac{\mathrm{d}X^1}{\mathrm{d}t} = \frac{1}{Lc}\frac{\mathrm{d}X^{12}}{\mathrm{d}t} = \sqrt{1 - \left(\frac{\kappa c^2}{p_0}\right)^2} \tag{69}$$

Thus to the energy of an object moving translationally at $\mathrm{d}X^1/\mathrm{d}t$ =1 m/s, there corresponds the 2-vector speed $\mathrm{d}X^{12}/\mathrm{d}t = L\,\mathrm{d}X^1/\mathrm{d}t = 1.6 \times 10^{-35}$ m^2/s (diameter speed 4 $\times 10^{-18}$m/s). This would be a typical 2-vector speed of a macroscopic object. For a microscopic object, such as the electron, which can be accelerated close to the speed of light, the corresponding 2-vector speed could be of the order of 10^{-26} m^2/s (diameter speed 10^{-13}m/s). In the examples above we have provided rough estimations of possible 2-vector speeds. Exact calculations should treat concrete situations of collisions of two or more objects, assume that not only 1-vector, but also 2-vector, 3-vector and 4-vector motions are possible, and take into account the conservation of the polyvector momentum P_A.

Maximum 1-vector speed, i.e., the usual speed, can exceed the speed of light when the holographic components such as $\mathrm{d}\sigma/\mathrm{d}t$, $\mathrm{d}X^{12}/\mathrm{d}t$, $\mathrm{d}X^{012}/\mathrm{d}t$, *etc., are different from zero.* This can be immediately verified from eqs. (65),(66). The speed of light is no longer such a strict barrier as it appears in the ordinary theory of relativity in M_4. In C-space a particle has extra degrees of freedom, besides the translational degrees of freedom. The scalar, σ, the bivector, X^{12} (in general, X^{rs}, $r, s = 1, 2, 3$) and the threevector, X^{012} (in general, X^{0rs}, $r, s = 1, 2, 3$), contributions to the C-space quadratic form (64) have positive sign, which is just opposite to the contributions of other components, such as X^r, X^{0r}, $X^{rst}, X^{\mu\nu\rho\sigma}$. Because the quadratic form has both + and − signs, the absolute value of the 3-velocity $\mathrm{d}X^r/\mathrm{d}X^0$ can be greater than c.

6 From the "block universe" in C-space to the relativistic dynamics and evolution in Minkowski spacetime

Extended relativity, i.e., the relativity in C-space, is based on the action (60). An equivalent action is

$$I[X^A, \lambda] = \frac{1}{2} \int \mathrm{d}\tau \left(\frac{\dot{X}^A \dot{X}_A}{\lambda} + \lambda \kappa^2 \right) \tag{70}$$

which is a functional of the variables $X^A \equiv X^{\mu_1 \dots \mu_r}$ and a scalar Lagrange multiplier λ. Variation of (70) with respect to λ gives

$$\dot{X}^A \dot{X}_A = \lambda^2 \kappa^2 \tag{71}$$

After inserting the latter relation into the action (70) we obtain the action (60), by which we have a confirmation that both actions are classically equivalent.

In the usual theory of relativity we often choose the parameter τ such that it is equal to the coordinate $X^0 \equiv t$. In the extended relativity we may choose parameter τ equal to the variable s of the pseudoscalar part of X (see eq.(28)), that is, we may put $\tau = s$. By doing, so we obtain a reduced theory in which all X^A, except s, are independent. So it was found [4] that the elegant Stueckelberg theory is naturally embedded in the C-space action (60) or (70). Here also lies a clue to a natural resolution of the "problem of time" in quantum gravity (see e.g. [10, 11]).

The discovery that we obtain the unconstrained Stueckelberg action in M_4 from the constrained action in C-space is so important that it is worth to discuss it here again. In refs. [4] I started from a yet another equivalent action, namely the phase space or first order action in C-space,

$$I[X^A, P_A, \lambda] = \int \mathrm{d}\tau \left(P_A \dot{X}^A - \frac{\lambda}{2}(P^A P_A - \kappa^2) \right) \tag{72}$$

which is a functional not only of the coordinate variables X^A, but also of the canonically conjugate momenta P_A. In this paper, instead, I will start first from the action (70) and then also directly from the action (60).

Let us now take the dimension of spacetime $n = 4$. Then the velocity polyvector reads

$$\dot{X} = \dot{X}^A E_A = \dot{\sigma} \underline{1} + \dot{x}^\mu \gamma_\mu + \tfrac{1}{2} \dot{x}^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \dot{\xi}^\mu I \gamma_\mu + \dot{s} I \tag{73}$$

Further, let us consider a special case in which

$$\dot{\sigma} = 0 \quad , \qquad \dot{x}^{\mu\nu} = 0 \quad , \qquad \dot{\xi}^{\mu} = 0 \tag{74}$$

so that the velocity polyvector and its square are simply

$$\dot{X} = \dot{x}^{\mu}\gamma_{\mu} + \dot{s}I \tag{75}$$

$$|\dot{X}|^2 = \dot{X}^A \dot{X}_A = \dot{x}^{\mu}\dot{x}_{\mu} - \dot{s}^2 \tag{76}$$

The actions (60) and (70) then assume the simplified forms

$$I[x^{\mu}, s] = \kappa \int \mathrm{d}\tau \, \sqrt{\dot{x}^{\mu}\dot{x}_{\mu} - \dot{s}^2} \tag{77}$$

and

$$I[x^{\mu}, s, \lambda] = \frac{1}{2} \int \mathrm{d}\tau \, \left(\frac{\dot{x}^{\mu}\dot{x}_{\mu} - \dot{s}^2}{\lambda} + \lambda\kappa^2 \right) \tag{78}$$

Let us first consider the Howe-Tucker-like [12] action (78). The corresponding equations of motion are

$$\delta x^{\mu} \; : \qquad \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{x}^{\mu}}{\lambda} \right) = 0 \tag{79}$$

$$\delta s \; : \qquad \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{s}}{\lambda} \right) = 0 \tag{80}$$

$$\delta \lambda \; : \qquad \sqrt{\dot{x}^{\mu} - \dot{s}^2} = \lambda\kappa \tag{81}$$

Using eq.(80) we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{s}s}{\lambda} \right) = \frac{\dot{s}^2}{\lambda} \tag{82}$$

Inserting the latter relation into the action (78) we find

$$I[x^{\mu}, s, \lambda] = \frac{1}{2} \int \mathrm{d}\tau \, \left[\frac{\dot{x}^{\mu}\dot{x}_{\mu}}{\lambda} + \lambda\kappa^2 - \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{s}s}{\lambda} \right) \right] \tag{83}$$

We will now use the fact that the Lagrange multiplier λ can be an arbitrary function of τ: choice of λ is related to choice of parametrization, i.e., choice of the parameter τ. Let us choose

$$\lambda = \Lambda \; , \; \text{ i.e. } , \qquad \sqrt{\dot{x}^{\mu}\dot{x}_{\mu} - \dot{s}^2} = \Lambda\kappa \tag{84}$$

where Λ is a fixed constant, and insert (84) into (83). Omitting the total derivative term (which does not influence the equations of motion) we obtain

$$I[x^{\mu}] = \frac{1}{2} \int \mathrm{d}\tau \, \left(\frac{\dot{x}^{\mu}\dot{x}_{\mu}}{\Lambda} + \Lambda\kappa^2 \right) \tag{85}$$

which is just the well known *Stueckelberg action* in which all x^μ are independent.

The equation of motion derived from the Stueckelberg action (85) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{x}^\mu}{\Lambda}\right) = 0 \tag{86}$$

and they are identical to the x^μ equations of motion (79) derived from the action (78) in the presence of the "gauge" condition (84). Eq. (86) implies that the momentum $p_\mu = \dot{x}_\mu/\Lambda$ and its square $p^\mu p_\mu$ are constant of motion.

The relation (84) can be written in the form

$$\sqrt{\mathrm{d}x^\mu \mathrm{d}x_\mu - \mathrm{d}s} = \Lambda\kappa\,\mathrm{d}\tau \tag{87}$$

which says that the parameter τ is equal to the length of the world line in C-space.

In order to further investigate the relation (84) let us consider the s equation of motion (80) from which we find

$$\frac{\kappa\dot{s}}{\sqrt{\dot{x}^\mu\dot{x}_\mu - \dot{s}^2}} = \frac{1}{C} \tag{88}$$

where C is a constant of integration. From (88) and (84) we have the relation

$$\frac{\Lambda}{C} = \frac{\mathrm{d}s}{\mathrm{d}\tau} \tag{89}$$

which says that the parameter τ is proportional to the variable s. By inserting (89) into the action (85) we find

$$I[x^\mu] = \frac{1}{2}\int \mathrm{d}s\,\left(\frac{1}{C}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x_\mu}{\mathrm{d}s} + C\kappa^2\right) \tag{90}$$

The evolution parameter in (90) is thus just the extra variable s entering the polyvector (73).

Instead of using the Howe-Tucker action (78) we can start directly from the action (77). The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\kappa\dot{x}^\mu}{\sqrt{\dot{x}^\mu\dot{x}_\mu - \dot{s}^2}}\right) = 0 \tag{91}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\kappa\dot{s}}{\sqrt{\dot{x}^\mu\dot{x}_\mu - \dot{s}^2}}\right) = 0 \tag{92}$$

which is equivalent to (79)–(81). The canonical momenta are

$$p_\mu = \frac{\kappa \dot{x}^\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu - \dot{s}^2}} \,, \qquad p_{(s)} = \frac{\kappa \dot{s}}{\sqrt{\dot{x}^\nu \dot{x}_\nu - \dot{s}^2}} \tag{93}$$

and they satisfy the constraint

$$p_\mu p^\mu - p_{(s)}^2 = \kappa^2 \tag{94}$$

Since $\dot{s} = \mathrm{d}s/\mathrm{d}\tau$, the action (77) can be written in the form

$$I[x^\mu] = \kappa \int \mathrm{d}s \sqrt{\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x_\mu}{\mathrm{d}s} - 1} \tag{95}$$

in which the parameter τ has disappeared and the variable s takes the role of the evolution parameter.

Alternatively, one can choose parameter τ in (77) such that $\dot{s} = 1$ (i.e., $\mathrm{d}s = \mathrm{d}\tau$) and we obtain the reduced action

$$I[x^\mu] = \kappa \int \mathrm{d}\tau \sqrt{\dot{x}^\mu \dot{x}_\mu - 1} \tag{96}$$

which is the same as (95).

Since the action (96) is *unconstrained* the components of the canonical momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{\kappa \dot{x}^\mu}{\dot{x}^\mu \dot{x}_\mu - 1} \tag{97}$$

are all independent. From the equations of motion it follows that p_μ is constant of motion and so is its square

$$p^\mu p_\mu = \frac{\kappa^2 \dot{x}^\mu \dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu - 1}} = M^2 \tag{98}$$

Here M^2 is *not* a fixed constant entering the action, but it is an arbitrary constant of motion.

The theory based on the unconstrained action (96) is equivalent to the Stueckelberg theory based on the unconstrained action (85). Whilst in the former theory gauge is chosen according to eq.(84), in the latter theory gauge is chosen according to $\dot{s} = 1$. Both theories are derived from the polyvector action (70), which—when eq.(74) holds—is equivalent to (77). Although (77) looks like an action which encompasses an extra dimension, this is not the case. In eq.(77) *s is not a variable due to an extra dimension, it is a coordinate of C-space, that is the Clifford manifold generated by a set of basis vectors spanning our spacetime.*

In C-space there is the *constraint*. Namely, the C-space momentum $P_A = \partial L / \partial X^A$ is constrained to the "mass shell""

$$P_A P^A = \kappa^2 \tag{99}$$

Explicitly, for our particular case (74) eq.(99) reads as eq.(94). That is, the four components p_μ and the extra component $p_{(s)}$ (due to the extra variable s) altogether are constrained. But the 4-momentum p_μ alone, i.e., the momentum belonging to the reduced action (85) or (96), is not constrained. And since (85) or (96) describe a free particle (without interaction), momentum p_μ is an arbitrary constant of motion.

The C-space action (70) or (77) is invariant under arbitrary reparametrizations of the parameter τ, and a consequence is the existence of the constraint. This is analogous to the situation in the ordinary theory of relativity. *Therefore in C-space we have a "block universe": everything is frozen once for all.*

The reduced actions (85) and (96) are not invariant under reparametrizations of τ, there is no constraint, and all four coordinates x^μ evolve independently along τ which is a true evolution parameter, identified in the case of eq.(85) with the length of the worldline in C-space or in the case of eq.(96) with the extra variable s. So we have evolution in 4-dimensional spacetime which is a section through C-space.

The enigma of why we feel the passage of time—a concept that does not exist in the theory of relativity—is thus resolved, at least formally (see also refs.[4, 10]), by postulating that spacetime slice M_4 moves through C-space. A consequence is a genuine dynamics (*relativistic dynamic*) on M_4. On the other hand, *all the elegance of relativity is preserved, not in spacetime, but in C-space.*

7 Conclusion

We have shifted the theory of relativity from the 4-dimensional spacetime to the Clifford space. The latter space is a very natural—in fact, unavoidable—generalization of spacetime V_4. The geometry of spacetime is described in terms of vectors which can be elegantly represented as Clifford numbers. Once we have a set of four Clifford numbers γ_μ representing four independent basis vectors of V_4 we can automatically generate a larger structure, namely the Clifford algebra $\mathcal{C}_4$ of V_4 which encompasses multivectors. Since $\mathcal{C}_4$ is a manifold, we call it the Clifford space or C-space. Owing to the fact, explored in this paper, that the extended object can be described by multivectors, it is very natural to assume that physics fundamentally takes place in C-space. This has many far reaching physical consequences, some of which are described in this paper (for the rest see also ref.[4]).

Relativity in C-space treats the multivector coordinates of an extended object as the *degrees of freedom* and predict that the associated multivector (holographic) velocities can be different from zero. In particular, when the bivector and higher multivector velocities are zero, then the new theory is indistinguishable from the existing theory of relativity. All the predictions of the ordinary theory of relativity are then preserved in the new theory. However, from the point of view of the relativity in C-space, the ordinary relativity has restricted validity: it holds only when $\dot{X}^{\mu\nu}$, $\dot{X}^{\mu\nu\alpha}$, etc., are zero. When they are different from zero we have violation of the ordinary relativity and Lorentz invariance (in M_4) in the sense that the worldlines of freely moving physical objetcs are no longer confined to M_4, but they escape into the larger space, namely C-space. The transformations that relate such worldlines are in general not the usual Lorentz transformations, but the generalized Lorentz transformations, namely the rotations in C-space (59). Lorentz group is thus no longer the exact symmetry group of physical objetcs and speed of light not the maximum speed. The possibility that Lorentz invariance might be violated has been widely discussed during last years [13] in various contexts, especially within noncommutative geometries [14].

Relativity in C-space implies that in Minkowski space M_4 there is the genuine dynamics—"relativistic dynamics"—as advocated by Stueckelberg and his followers [5]. The Stueckelberg theory, since being unconstrained, has many desirable properties, especially when quantized [5, 15] and provides a natural explanation [4, 10, 16] of "the passage of time", whilst the more general theory, namely the relativity in C-space retains all the nice features and elegance of the theory of relativity, including reparametrization invariance, "block universe", etc.. The objects, when viewed from the C-space perspective are infinitely extended, frozen, "world lines" $X^A(\tau) \equiv X^{\mu_1 \dots \mu_r}(\tau)$, analogous to the world lines of the ordinary relativity. When viewed from the perspective of 4-dimensional spacetime (which is just a subspace of the "full" space, namely C-space) the objects have *finite* extensions both in space-like and time-like direction. They move in spacetime, and during the motion their extensions, including the time-like extensions, change with τ.

From the point of view of the new theory, p-branes (including strings)—whose $(p+1)$-dimensional worldsheets are infinitely extended objects along time-like directions— are merely idealized objects to which the real physical objects approach when the evolution parameter τ goes to infinity. At finite τ, all objects in spacetime are predicted to have finite extension.

All this is just the start. A Pandora box of fascinating new possibilities is open, some of them are touched in a recent book [4] and refs. [17]. A very promising perspective is in exploiting the well known fact [18] that spinors are members of left and right minimal ideals of Clifford algebra. This means that spinors are nothing but special kind of polyvectors. Therefore the presence of spinors is automatically included in the coordinate polyvector $X = X^A E_A$. We therefore expect that suitable generalizations of the action (60) (see [4]) will provide a description of spinning p-branes and super p-branes including superstrings, superparticles or spinning strings and spinning particles. It seems very likely that further development of the theory based on C-space will lead us towards M-theory and towards the unified theory of the known fundamental interactions.

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